北京航空航天大學
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Chapter 6
Free Electron Fermi Gas

6.1 Electron Gas Model and its Ground State

6.2 Thermal Properties of Electron Gas

6.3 Free Electrons in Electric Fields

6.4 Hall Effect

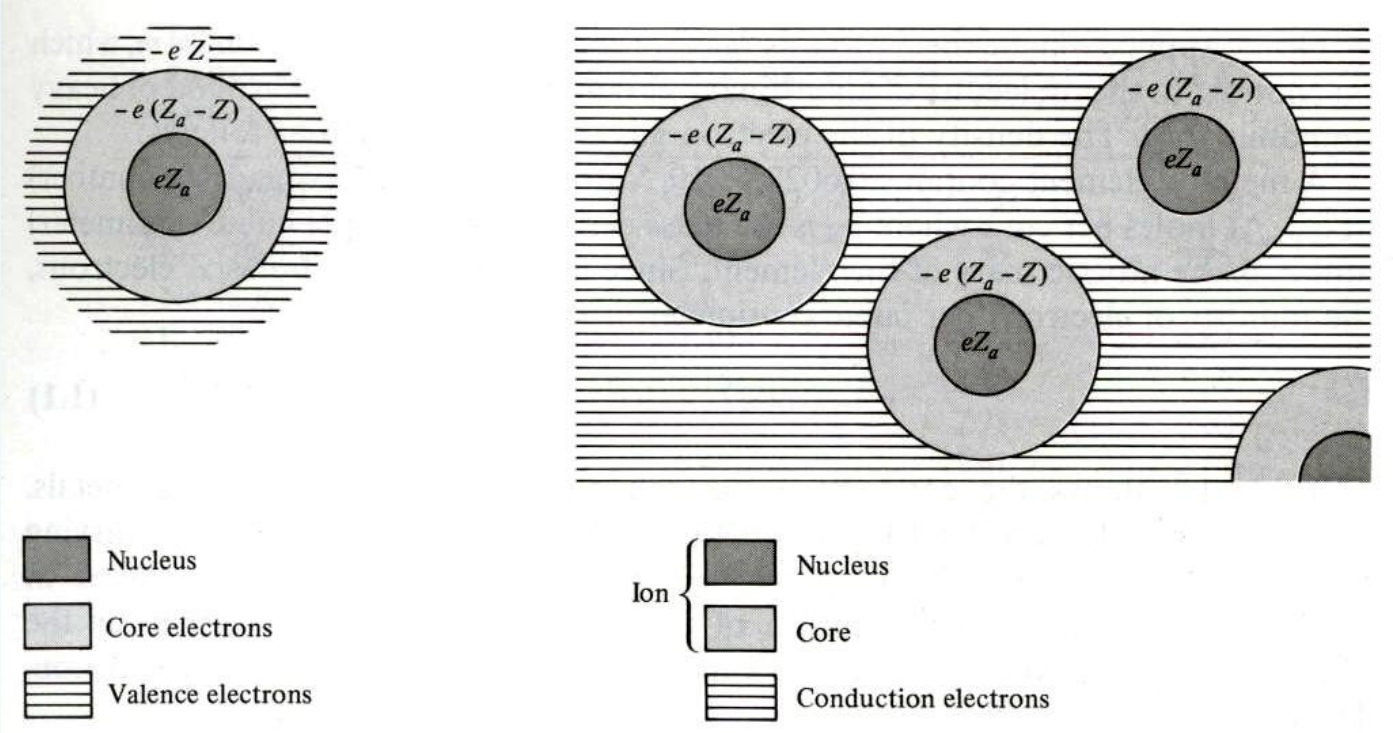
6.5 Thermal Conductivity of Metals

6.6 Failures of the free electron gas model

6.1 Electron Gas Model and its Ground State

I. Basic Assumptions of Electron Gas Model

Metal: valence electrons → conduction electrons (moving freely)



✓ The **simplest** metals are the **alkali** metals—**lithium**, **sodium**, **potassium**, **cesium**, and **rubidium**.

6.1 Electron Gas Model and its Ground State

density of electrons:

$$n = N_A \frac{Z\rho_m}{A}$$

where Z *is* # of conduction electrons per atom, A is relative atomic mass, ρ_m is the density of mass in the metal. The spherical volume of each electron is,

$$\frac{1}{n} = \frac{V}{N} = \frac{4}{3}\pi r_s^3 \qquad r_s = \left(\frac{3}{4\pi n}\right)^{1/3}$$

Ordinary metals:
$n \sim 10^{28}/m^3$
$r_s \sim 10^{-9}\, m$

Free electron gas model: Suppose, except the confining potential near surfaces of metals, conduction electrons are completely free. The conduction electrons thus behave just like gas atoms in an ideal gas --- free electron gas.

Basic Properties：

✓ **Ignore interactions of electron-ion type (free electron approx.)**

✓ **And electron-eletron type (independent electron approx).**

⇒ **Total energy are of kinetic type,**

ignore potential energy contribution.

✓ **The classical theory had several conspicuous successes**

Long Mean Free Path:

- ✓ From many types of experiments it is clear that a conduction electron in a metal can *move freely in a straight path* over *many atomic distances*.
- ✓ In a very pure specimen at *low temperatures*, the mean free path may be as long as 10^8 interatomic spacings (more than 1 cm).
- ✓ Condensed matter so transparent to conduction electrons:
- Due to periodic lattice structure.
- Due to quantum nature of electrons: Pauli exclusive principle.

II. Quantum Free Electron Model -- single electron state and eigenenergy

1. Schrödinger Equation

⇒ **Single electron problem in a 3D space**

Equation of wave for each electron (*within* the potential well) :

$$-\frac{\hbar^2}{2m}\nabla^2\psi(\vec{r})=\varepsilon\psi(\vec{r})$$

$$-\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial^2 x}+\frac{\partial^2}{\partial^2 y}+\frac{\partial^2}{\partial^2 z}\right)\psi(x,y,z)=\varepsilon\psi(x,y,z)$$

2. Boundary Condition ——periodic B.C.

$$\psi(x+L,y,z,t)=\psi(x,y,z,t)$$
$$\psi(x,y+L,z,t)=\psi(x,y,z,t)$$
$$\psi(x,y,z+L,t)=\psi(x,y,z,t)$$

3. Solution of Schrödinger's equation

$$\psi_{\vec{k}}(\vec{r})=\frac{1}{\sqrt{V}}e^{i\vec{k}\cdot\vec{r}}$$

$\vec{k}$ --wave vector, $\vec{r}$ --position vector

$$\vec{k}=(k_x,k_y,k_z)$$

energy eigenvalues：

$$\varepsilon(\vec{k})=\frac{\hbar^2k^2}{2m}=\frac{\hbar^2(k_x^2+k_y^2+k_z^2)}{2m}$$

4. physical meaning of $\vec{k}$ vector

$$-i\hbar\nabla\psi_{\vec{k}}(\vec{r})=\hbar\vec{k}\psi_{\vec{k}}(\vec{r})$$

$\hbar\vec{k}$: eigen value

$\hat{p}=-i\hbar\nabla$ --momentum operator

Electron in state $\psi_{\vec{k}}(\vec{r})$ has a certain momentum proportional to wave vector $\vec{k}$:

$$\vec{p}=\hbar\vec{k}$$

velocity: $\vec{v}=\dfrac{\vec{p}}{m}=\dfrac{\hbar\vec{k}}{m}$

energy: $\varepsilon=\dfrac{p^2}{2m}=\dfrac{1}{2}mv^2$

wavelength : $\lambda=\dfrac{2\pi}{k}$

Require the plain wave solutions to fullfil PBC:

$$e^{ik_x L} = e^{ik_y L} = e^{ik_z L} = 1 \quad \Rightarrow \quad k_x = \frac{2\pi}{L} n_x \qquad k_y = \frac{2\pi}{L} n_y \qquad k_z = \frac{2\pi}{L} n_z$$

n_x, n_y, n_z —— **a series of integer numbers.**

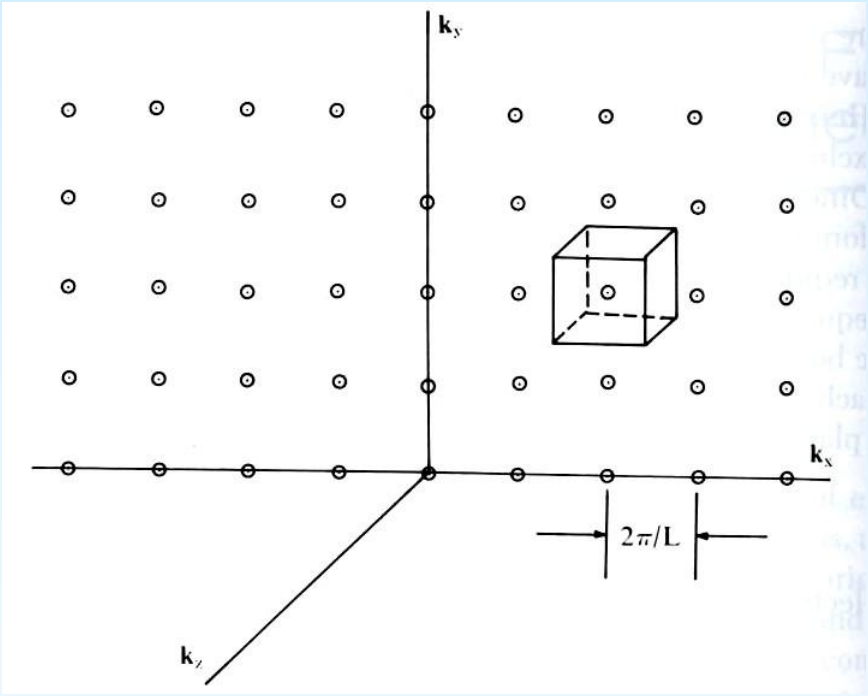
Free electron's eigen energy levels are discontineous, and are pretty close to each other.

5. *k*-space and density of states (DOS)

In the k-space, every $\vec{k}$ point represents a possible state of electron (orbital). Neigboring representative points are equally spaced by $2\pi/L$ in three dimensions. So the volume each point occupies is:

$$\Delta\vec{k} = \left(\frac{2\pi}{L}\right)^3 = \frac{8\pi^3}{V}$$

6.1 Electron Gas Model and its Ground State



$$\Delta\vec{k} = \left(\frac{2\pi}{L}\right)^3 = \frac{8\pi^3}{V}$$

Unit *k*-space volume

contains # of $\vec{k}$ points:

$$\frac{1}{\Delta k} = \frac{V}{8\pi^3}$$

For each $\vec{k}$ state (orbital), electrons have two different spin states: unit $\vec{k}$ space volthus can ume contain at most $V/4\pi^3$ individual spinful electronic states :

DOS in $\vec{k}$ space: $g(\vec{k})$

$$\mathrm{d}N = g(\vec{k})\,\mathrm{d}\tau_{\vec{k}} = \frac{V}{4\pi^3}\mathrm{d}\tau_{\vec{k}}$$

Densisy of states in energy scale $g(\varepsilon)$:

unit *volume*, unit *energy invterval*, *spinful electronic states*:

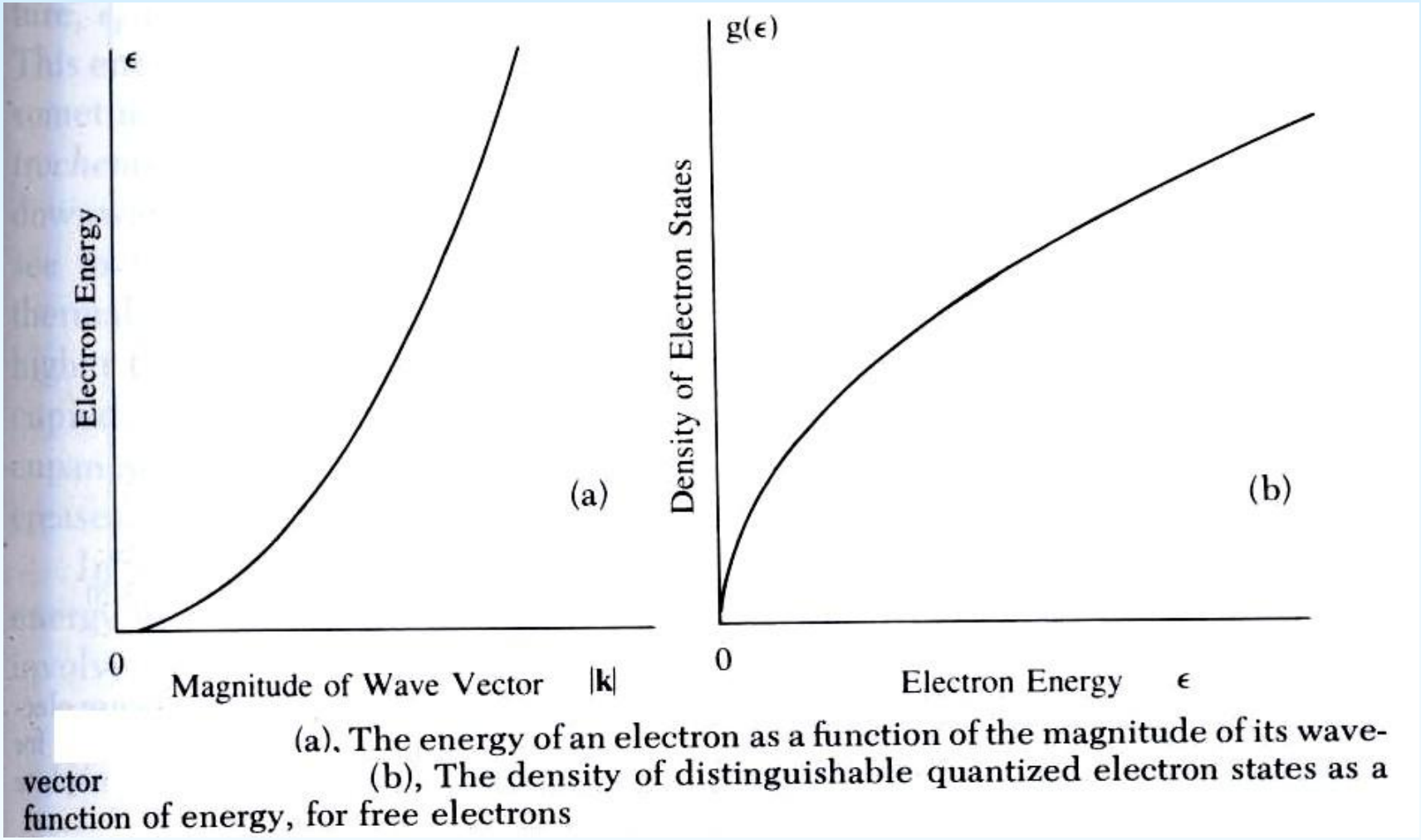
$$\mathrm{d}N = Vg(\varepsilon)\mathrm{d}\varepsilon$$

$\varepsilon = \hbar^2 k^2/2m$ determines the isosurface in $\vec{k}$ space, which turns out to be spherical surfaces with radius $\sqrt{2m\varepsilon}/\hbar$.

Electronic states between energy shells $\varepsilon \sim \varepsilon + \mathrm{d}\varepsilon$ corresponds to those with wave vectors between $k \sim k + \mathrm{d}k$

$$\mathrm{d}N = \frac{V}{4\pi^3}4\pi k^2\mathrm{d}k \quad \Rightarrow \quad g(\varepsilon) = \frac{1}{\pi^2\hbar^3}\left(2m^3\varepsilon\right)^{1/2}$$

6.1 Electron Gas Model and its Ground State



(a). The energy of an electron as a function of the magnitude of its wave-vector
(b), The density of distinguishable quantized electron states as a function of energy, for free electrons

III. Ground state properties of free electron gas

Ground state of free electron gas means its property at the absolutely zero temperature.

Fermi sphere: obeying Pauli principle, electrons fills the orbitals from low energy to high levels, and form a sphere in *k* space.

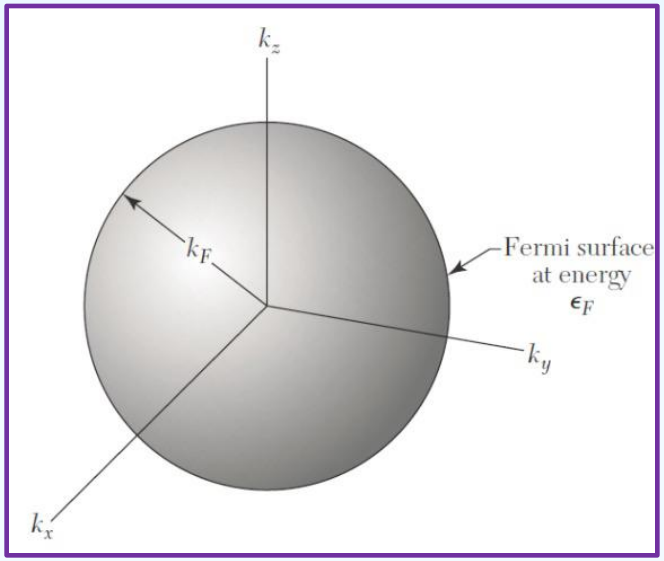
Fermi surface: Interface betwee occupied and unoccupied states

$$\epsilon_F = \frac{\hbar^2}{2m} k_F^2$$

Fermi wave vector: $\vec{k}_F$

$$2 \cdot \frac{4\pi k_F^3/3}{(2\pi/L)^3} = \frac{V}{3\pi^2} k_F^3 = N$$

$$k_F = \left(3\pi^2 n\right)^{1/3}$$



6.1 Electron Gas Model and its Ground State

Fermi energy: Single electron energy on the Fermi surface

$$\varepsilon_F = \frac{\hbar^2 k_F^2}{2m} \Rightarrow \boxed{\epsilon_F = \frac{\hbar^2}{2m}\left(\frac{3\pi^2 N}{V}\right)^{2/3}} \Leftrightarrow \boxed{N = \frac{V}{3\pi^2}\left(\frac{2m\epsilon}{\hbar^2}\right)^{3/2}}$$

Fermi velocity: The electron velocity v_F at the Fermi surface

$$\boxed{v_F = \left(\frac{\hbar k_F}{m}\right) = \left(\frac{\hbar}{m}\right)\left(\frac{3\pi^2 N}{V}\right)^{1/3}}$$

Fermi temperature: T_F is defined as ε_F/k_B

Has nothing to do with the real temperature of the electron gas!

Excise: mean energy of electrons：

$$\bar{\varepsilon} = \frac{1}{N}\int_0^{\varepsilon_F} \varepsilon g(\varepsilon)\mathrm{d}\varepsilon$$

$$= \int_0^{\varepsilon_F} \varepsilon g(\varepsilon)\mathrm{d}\varepsilon \Big/ \int_0^{\varepsilon_F} g(\varepsilon)\mathrm{d}\varepsilon$$

$$\bar{\varepsilon} = \frac{3}{5}\varepsilon_F = \frac{3}{5}k_B T_F$$

Even the ground state has some finite kinetic energy

--- due to Pauli principle

6.2 Thermal Properties of Free Electron Gas

I. Fermi Distribution

Distribution of free electrons in energy levels: Fermi-Dirac statistics

At temperature T, the probability that energy level ε_i is occupied in thermal equilibrium:

$$f_i = \frac{1}{e^{(\varepsilon_i-\mu)/k_BT}+1}$$

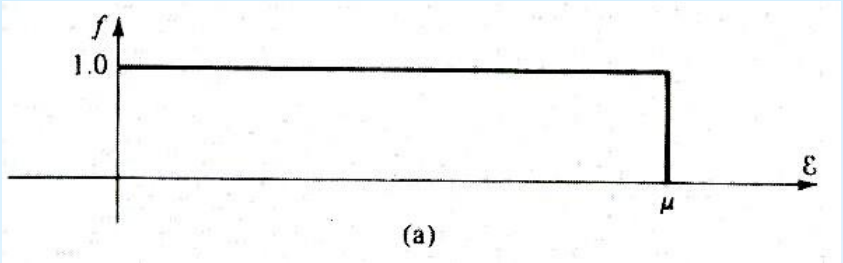
μ-**chemical potential**: in the condition of fixed temperature and volume, the increasement of free energy by adding a singlet electron to the system.

$$N = \sum_i f_i = V\int_0^\infty g(\varepsilon)f(\varepsilon)\mathrm{d}\varepsilon$$

6.2 Thermal properties of free electron gas

1. $T \to 0$

$$\begin{cases} \lim\limits_{T\to 0} f(\varepsilon_i) = 1, & \varepsilon_i < \mu \\ \lim\limits_{T\to 0} f(\varepsilon_i) = 0, & \varepsilon_i > \mu \end{cases}$$

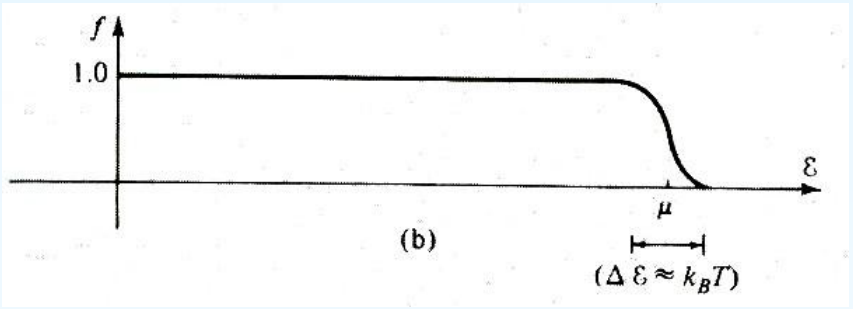


At absolutely zero temperature, levels with energy < μ are *all occupied*, and the levels above it are *all empty*. Chemical potential (Fermi energy) is the highest energy level electrons takes at *T=0*;

2. $T > 0$

$$\varepsilon = \mu \qquad f(\varepsilon) = \frac{1}{2}$$

μ level have equal probability to be filled or to ramain empty.



T rises, the step becomes more and more blurred.

$$\varepsilon << \mu \quad f(\varepsilon) = 1$$

$$\varepsilon >> \mu \quad f(\varepsilon) = 0$$

II. Chemical Potential

1. ground state $T=0,\ \varepsilon_F$

$$\mu(0)=\varepsilon_F=\frac{\hbar^2k_F^2}{2m}=\frac{\hbar^2}{2m}\left(3n\pi^2\right)^{\frac{2}{3}}$$

2. thermal excitation $T>0 \quad k_BT\ll\mu$

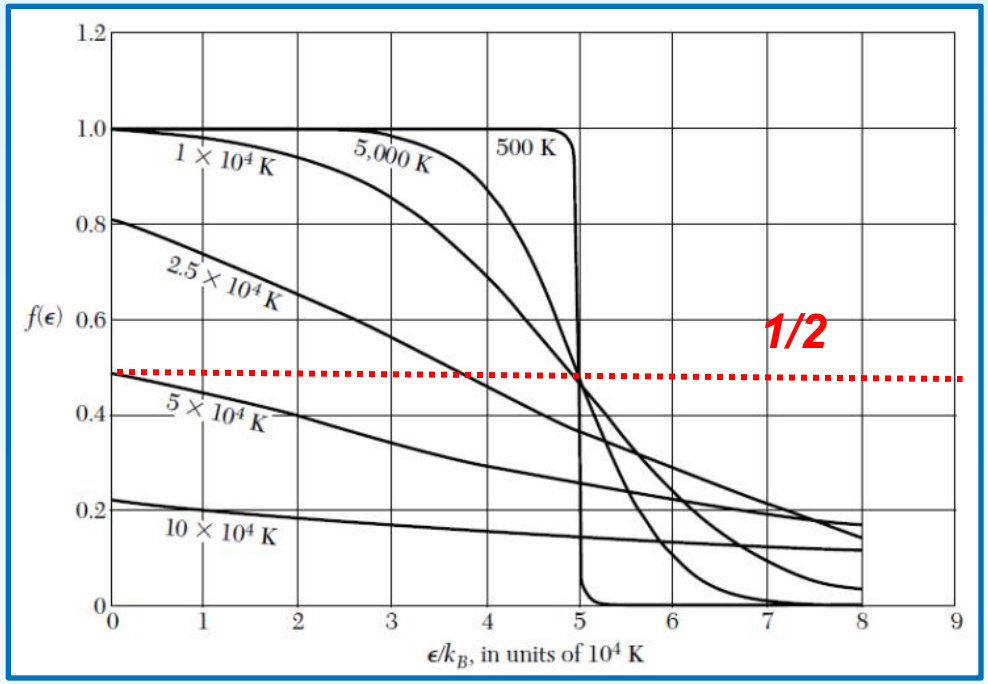
Electrons **near Fermi surface can be excited to states outside Fermi surface, leaving some empty states (*holes*) inside.**

$$N=V\int_0^\infty g(\varepsilon)f(\varepsilon)\mathrm{d}\varepsilon \quad \Leftarrow \quad g(\varepsilon)=\frac{1}{\pi^2\hbar^3}\left(2m^3\varepsilon\right)^{1/2}$$

$$\boxed{\mu=\varepsilon_F\left[1-\frac{\pi^2}{12}\left(\frac{k_BT}{\varepsilon_F}\right)^2\right]}$$ *generally,* $k_BT\ll\varepsilon_F \quad \mu\approx\varepsilon_F$

Fermi-Dirac distribution function

$$f(\epsilon) = \frac{1}{\exp[(\epsilon - \mu)/k_B T] + 1} .$$



Valency	Metal	Fermi temperature $T_F = \epsilon_F/k_B$, in deg K
1	Li	5.48×10^4
	Na	3.75
	K	2.46
	Rb	2.15
	Cs	1.83
	Cu	8.12
	Ag	6.36
	Au	6.39
2	Be	16.41
	Mg	8.27
	Ca	5.43
	Sr	4.58
	Ba	4.24
	Zn	10.90
	Cd	8.66
3	Al	13.49
	Ga	12.01
	In	9.98
4	Pb	10.87
	Sn(*w*)	11.64

- ✓ Fermi temperature is high: T_F= 50,000 K.
- ✓ The ***chemical potential*** at each temperature may be read off the graph as the energy at which f = 0.5.

3. Density of States

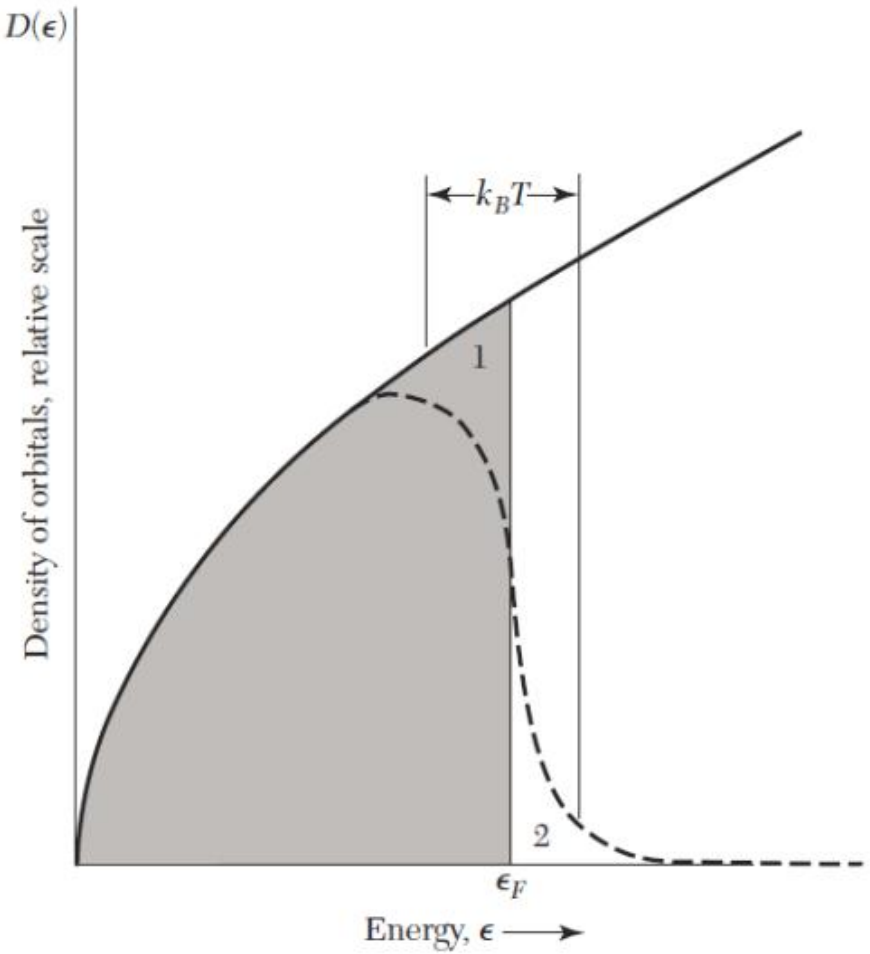


Figure 5 Density of single-particle states as a function of energy, for a free electron gas in three dimensions. The dashed curve represents the density $f(\epsilon, T)D(\epsilon)$ of filled orbitals at a finite temperature, but such that k_BT is small in comparison with ϵ_F. The shaded area represents the filled orbitals at absolute zero. The average energy is increased when the temperature is increased from 0 to T, for electrons are thermally excited from region 1 to region 2.

III.electron heat capacity

contant volume specific heat : $c_V = \left(\frac{\partial u}{\partial T}\right)_V \qquad u = \frac{U}{V}$

***U*—energy :** $U = 2\sum_{\vec{k}} \varepsilon(\vec{k}) f_{\vec{k}} \qquad N = 2\sum_{\vec{k}} f_{\vec{k}}$

$$u = \int_0^\infty \varepsilon g(\varepsilon) f(\varepsilon) \mathrm{d}\varepsilon \qquad n = \int_0^\infty g(\varepsilon) f(\varepsilon) \mathrm{d}\varepsilon$$

$$u = u_0 + \frac{\pi^2}{6}[(k_B T)^2 g(\varepsilon_F)]$$

u_0 --average energy at ground state

$$c_V = \left(\frac{\partial u}{\partial T}\right)_V = \frac{\pi^2}{2} n k_B \frac{T}{T_F}$$

6.2 Thermal properties of free electron gas

Analysis : classically, mean energy of *1* mol electron gas

$$\bar{\varepsilon}_{mol} = N_A\left(\frac{3}{2}k_BT\right) = \frac{3}{2}RT$$

monovalent metal : $C^e_{V,mol} = \dfrac{\partial \bar{\varepsilon}_{mol}}{\partial T} = \dfrac{3}{2}R$

high *T* total specific heat (constant volume) of metal:

$$C_V = C^{Ph}_{V,mol} + C^e_{V,mol} = 3R + \frac{3}{2}R \approx 37.40\,\mathrm{J/mol\cdot K}$$

actual $C^e_{V,\ mol}$ is smaller than this classical limit

quantum : $C^e_V \sim \dfrac{T}{T_F}$

At room temperatures, electrons' contribution to total specific heat is *negligible*.

6.2 Thermal properties of free electron gas

$$C_V^e = \gamma T \qquad \gamma \text{ is elctron specific heat constant}$$

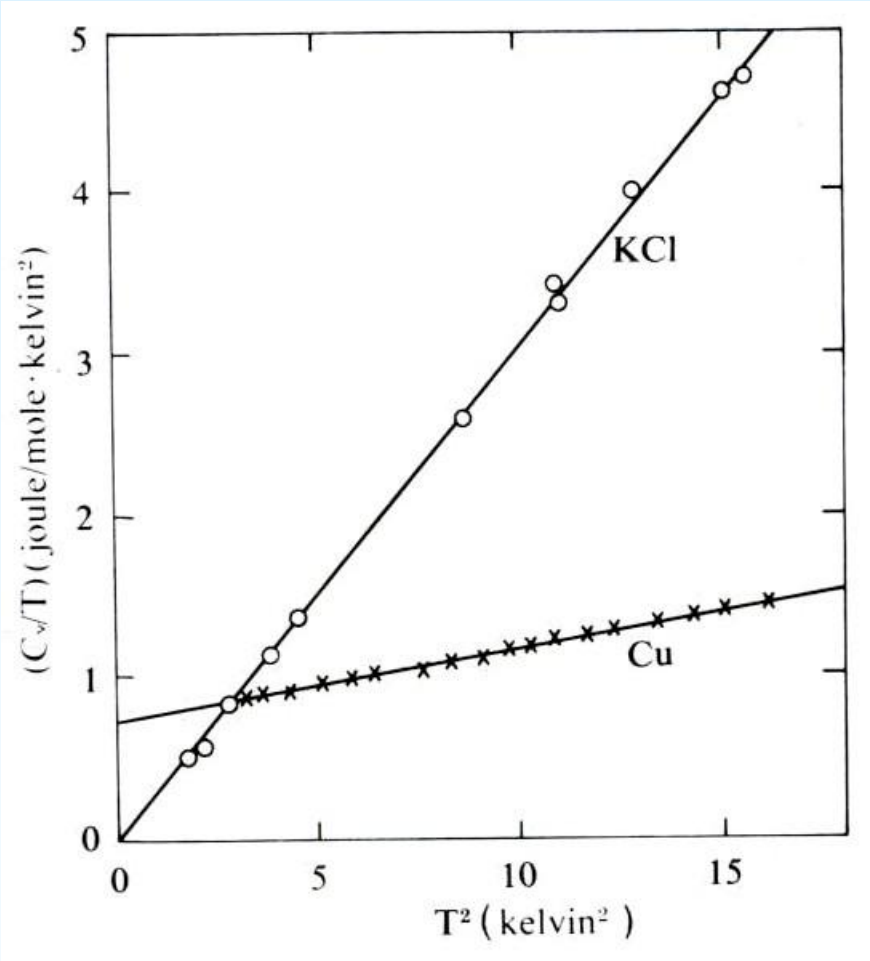
low T : $$C_{V,mol}^{Ph} = \frac{12}{5}\pi^4 R\left(\frac{T}{\Theta_D}\right)^3$$

$$C_V = C_{V,mol}^e + C_{V,mol}^{Ph}$$

$$= \gamma T + \beta T^3$$

$$\frac{C_V}{T} = \gamma + \beta T^2$$

$$\frac{C_V}{T} \sim T^2 \text{ ——quadratic}$$



Thermal Effective Mass

$$C/T = \gamma + AT^2$$

Thermal effective mass m_{th} to the electron mass m:

$$\frac{m_{\text{th}}}{m} \equiv \frac{\gamma(\text{observed})}{\gamma(\text{free})}$$

Due to ***3 facts***: interaction with ***periodic potential***, with ***phonon***, with ***electrons*** themselves.

Heavy Fermions

Several metallic compounds have been discovered that have ***enormous values***, **two or three orders** of magnitude higher than usual, of the electronic heat capacity constant.

6.3 Free Electrons in Electric Fields

Transport properties of electrons: *electric* & *thermal* *conductivity*.

I. Drude model for free electrons

- ✓ Drude's assumptions: neglects any long-range interaction between the ***electron*** and the ***ions*** or between the ***electrons***.
- ✓ Consider only the ***instantaneous*** collisions.

➢ **relaxation time τ**

Paul Drude
German physicist

Probability of collision within the time intervel $\mathbf{d}t$:

$$\mathbf{d}t/\tau$$

$$\Delta x \approx \frac{\hbar}{\Delta p} >> \frac{1}{k_F} \approx r_s$$ of the order of 1 Å ⇒ **Semiclassical**

II. Electron Gas in External Fields: Equation of Motion

At moment *t*, *average momentum* of electrons $\vec{p}(t)$, external force *F*(*t*). After time d*t*, the chance that an electrons is uninvolved in any collision is (1-d*t*/τ), their *contribution* to mean momentum is:

$$t+\mathrm{d}t: \qquad \vec{p}(t+\mathrm{d}t)=\left(1-\frac{\mathrm{d}t}{\tau}\right)\left[\vec{p}(t)+\vec{F}(t)\mathrm{d}t\right]$$

$$\vec{p}(t+\mathrm{d}t)-\vec{p}(t)=\vec{F}(t)\mathrm{d}t-\vec{p}(t)\frac{\mathrm{d}t}{\tau}$$

$$\Rightarrow \quad \boxed{\frac{\mathrm{d}\vec{p}(t)}{\mathrm{d}t}=\vec{F}(t)-\frac{\vec{p}(t)}{\tau}}$$

——E.O.M. of electron gas in external fields

III. Electric Conductivity of Metals

1. Classical Picture

In the *absence* of electric fields, conduction electrons are in a random motion:

$$\vec{v}_{\text{av}} = 0$$

External Fields are *on*：conduction electrons are drifting towards a specific direction, with a (drifting) velocity $\vec{v}_{\text{d}}(t)$.

① since $$m\frac{\mathrm{d}\vec{v}_{\text{d}}(t)}{\mathrm{d}t} = \vec{F}(t) - m\frac{\vec{v}_{\text{d}}(t)}{\tau}$$

In a Steady State:

$$\vec{F} = -e\vec{E} \qquad \mathrm{d}v_d(t)/\mathrm{d}t = 0$$

$$\Rightarrow \quad \vec{v}_{\text{d}} = -\frac{e\vec{E}\tau}{m}$$

② For any single electron, since its last collision, in time *t* it travels:

in the absence of fields : $\vec{v}_0$

if electric field is on: $-eEt/m$

Suppose that the electrons move radomly after collisions

$\vec{v}_0$ has nothing to do with average velocity of the electron gas $\vec{v}_{\mathrm{av}}$

$\vec{v}_{av}$ is determined by drift velocity of each electron:

$$\vec{v}_{\mathrm{d}} = \vec{v}_{\mathrm{av}} = -\frac{e\vec{E}\tau}{m}$$

6.3 Free Electrons in Electric Fields

electric current : $\vec{J} = -ne\vec{v}_{\mathrm{d}} = \dfrac{ne^2\tau}{m}\vec{E}$ $\sigma = \dfrac{ne^2\tau}{m}$

$\Rightarrow$ $\boxed{\vec{J} = \sigma\vec{E}}$ --Ohm's law

✓ Electric conductivity ∝ density of conduction electrons, *inversly proportional* to electron mass *m*.

6.3 Free Electrons in Electric Fields

Table 3 Electrical conductivity and resistivity of metals at 295 K

(Resistivity values as given by G. T. Meaden, *Electrical resistance of metals*, Plenum, 1965; residual resistivities have been subtracted.)

Conductivity in units of 10^5 (ohm-cm)$^{-1}$.
Resistivity in units of 10^{-6} ohm-cm.

Li	Be											B	C	N	O	F	Ne
1.07	3.08																
9.32	3.25																
Na	Mg											Al	Si	P	S	Cl	Ar
2.11	2.33											3.65					
4.75	4.30											2.74					
K	Ca	Sc	Ti	V	Cr	Mn	Fe	Co	Ni	Cu	Zn	Ga	Ge	As	Se	Br	Kr
1.39	2.78	0.21	0.23	0.50	0.78	0.072	1.02	1.72	1.43	5.88	1.69	0.67					
7.19	3.6	46.8	43.1	19.9	12.9	139.	9.8	5.8	7.0	1.70	5.92	14.85					
Rb	Sr	Y	Zr	Nb	Mo	Tc	Ru	Rh	Pd	Ag	Cd	In	Sn (w)	Sb	Te	I	Xe
0.80	0.47	0.17	0.24	0.69	1.89	~0.7	1.35	2.08	0.95	6.21	1.38	1.14	0.91	0.24			
12.5	21.5	58.5	42.4	14.5	5.3	~14.	7.4	4.8	10.5	1.61	7.27	8.75	11.0	41.3			
Cs	Ba	La	Hf	Ta	W	Re	Os	Ir	Pt	Au	Hg liq.	Tl	Pb	Bi	Po	At	Rn
0.50	0.26	0.13	0.33	0.76	1.89	0.54	1.10	1.96	0.96	4.55	0.10	0.61	0.48	0.086	0.22		
20.0	39.	79.	30.6	13.1	5.3	18.6	9.1	5.1	10.4	2.20	95.9	16.4	21.0	116.	46.		
Fr	Ra	Ac															

Ce	Pr	Nd	Pm	Sm	Eu	Gd	Tb	Dy	Ho	Er	Tm	Yb	Lu
0.12	0.15	0.17		0.10	0.11	0.070	0.090	0.11	0.13	0.12	0.16	0.38	0.19
81.	67.	59.		99.	89.	134.	111.	90.0	77.7	81.	62.	26.4	53.
Th	Pa	U	Np	Pu	Am	Cm	Bk	Cf	Es	Fm	Md	No	Lr
0.66		0.39	0.085	0.070									
15.2		25.7	118.	143.									

Arnold Sommerfeld

Arnold Sommerfeld
German Physicist
1868-1951

In solid-state physics, the ***free electron model*** is a simple model for the behaviour of valence electrons in a crystal structure of a metallic solid.

It was developed principally by ***Arnold Sommerfeld*** who ***combined*** the ***classical Drude model*** with ***quantum mechanical Fermi–Dirac statistics*** and hence it is also known as the **Drude–Sommerfeld model**.

Doctoral students: Werner Heisenberg, Wolfgang Pauli, Peter Debye, Paul Sophus Epstein, Hans Bethe, Ernst Guillemin, Karl Bechert, Paul Peter Ewald, Herbert Fröhlich, Erwin Fues, Helmut Hönl, Ludwig Hopf, Walther Kossel, Adolf Kratzer, Alfred Landé, Otto Laporte, Wilhelm Lenz, Rudolf Peierls, Walter Rogowski, Rudolf Seeliger, Heinrich Welker, Gregor Wentzel

Other notable students: Herbert Kroemer, Linus Pauling, Walter Heitler

2. Sommerfeld's Picture

free electrons : $m\vec{v} = \hbar\vec{k}$

Arnold Sommerfeld
German Physicist
1868-1951

under a uniform and time-independent electric fields

$$\vec{F} = -e\vec{E}$$

6.3 Free Electrons in Electric Fields

according to Newton's second law：

$$\vec{F} = m\frac{\mathrm{d}}{\mathrm{d}t}\vec{v} = \hbar\frac{\mathrm{d}}{\mathrm{d}t}\vec{k} = -e\vec{E}$$

integrate it over time, we get

$$\vec{k}(t) - \vec{k}(0) = -\frac{e}{\hbar}\vec{E}t$$

In the absence of collisions, steady electric fields drives the Fermi sphere in $\vec{k}$ space to move in a constant velocity.

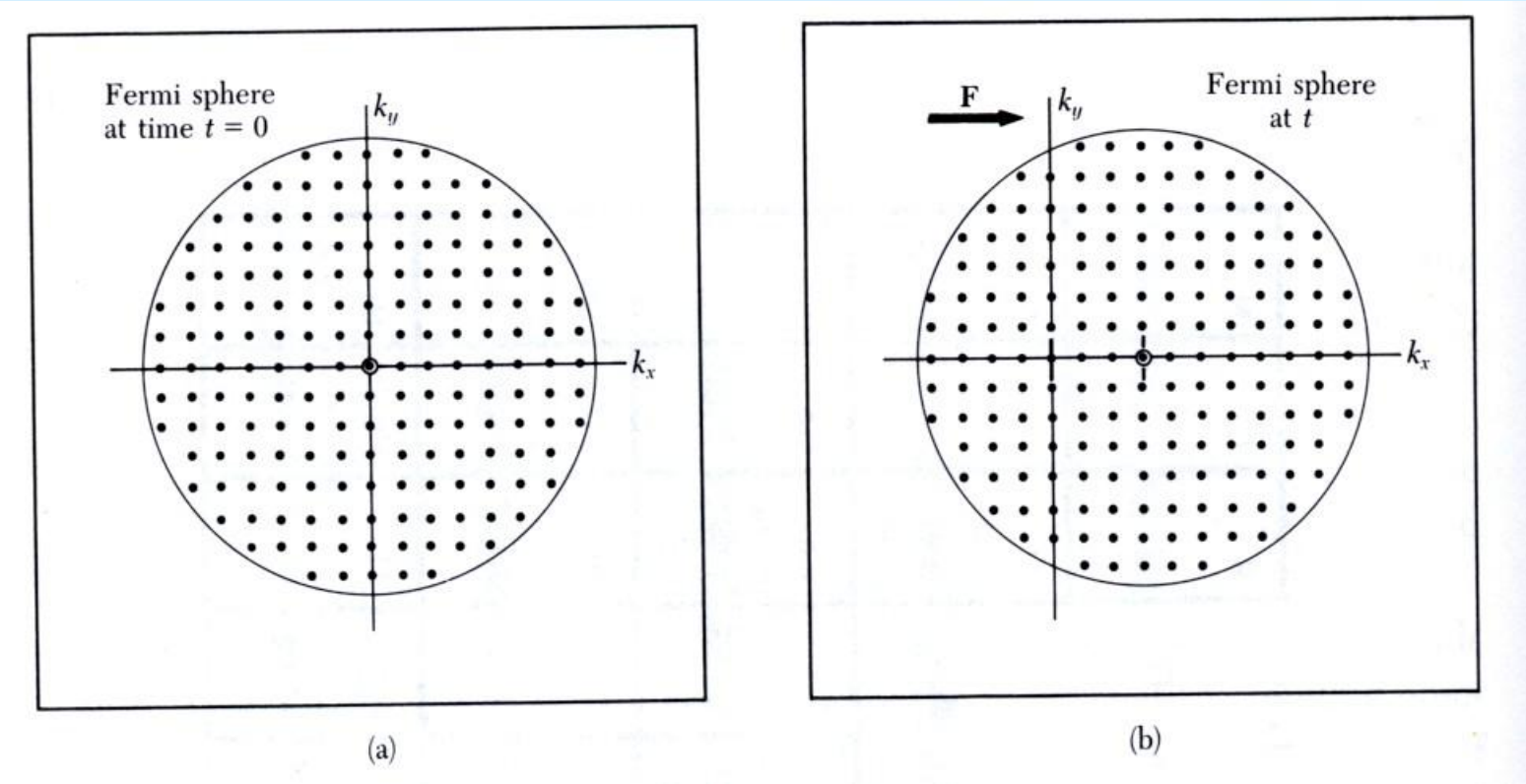
Electron gas fills Fermi sphere in *k* space centered at $\vec{k}$.

$t = 0$ apply electric field $\vec{E}$

at time *t*, the center of Fermi sphere moves to:

$$\delta\vec{k} = -e\vec{E}t/\hbar$$

6.3 Free Electrons in Electric Fields



The scattering of electrons with *impurities*, *crystal defects*, as well as *phonons*, keeps the moving Fermi sphere in a steady state.

Give *relaxation* time τ, the center of Fermi sphere moves：

$$\delta\vec{k} = -e\vec{E}\tau/\hbar$$

$$\vec{v}_d = -\frac{e\vec{E}\tau}{m}$$

$$\Rightarrow \quad \vec{J} = -ne\vec{v}_d = \frac{ne^2\tau}{m}\vec{E}$$

$$\sigma = \frac{ne^2\tau}{m} \qquad \boxed{\vec{J} = \sigma\vec{E}}$$

3. Mean Free Path of Electrons

classical electron gas : $l = v_0\tau$

quantum electron gas : $l = v_F\tau$

MFP of electrons is much larger than that classical theory predicts.

6.3 Free Electrons in Electric Fields

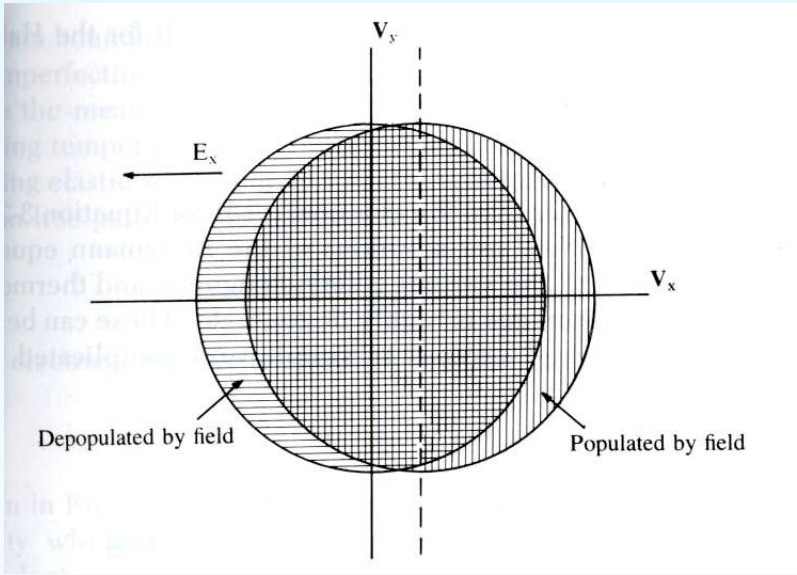
classical electron gas— resistivity is due to collisions between electrons and electrons, lattice ions, etc.

quantum electron gas—when ideal periodic structure is broken, collision with atomic vibration, defects contributes to resistivity.

Fermi Statistics & Band theory :

$$\sigma = \frac{ne^2\tau_F}{m^*}$$

- ✓ **Electric current is contributed by *a small portion* of electrons moving with *large velocity*.**



6.4 Hall Effect

The force an electron feels in a magnetic field：

$$\vec{F} = -e\left(\vec{E} + \vec{v} \times \vec{B}\right)$$

Equation of Motion：

$$\frac{\mathrm{d}\vec{p}(t)}{\mathrm{d}t} = -e\left(\vec{E} + \vec{v} \times \vec{B}\right) - \frac{\vec{p}(t)}{\tau} \qquad \vec{p} = m\vec{v}$$

If $\vec{B} // z$, equation changes as:

$$\begin{cases} m\left(\dfrac{\mathrm{d}}{\mathrm{d}t} + \dfrac{1}{\tau}\right)v_x = -e\left(E_x + v_y B\right) \\ m\left(\dfrac{\mathrm{d}}{\mathrm{d}t} + \dfrac{1}{\tau}\right)v_y = -e\left(E_y - v_x B\right) \\ m\left(\dfrac{\mathrm{d}}{\mathrm{d}t} + \dfrac{1}{\tau}\right)v_z = -eE_z \end{cases}$$

6.4 Hall Effect

For steady current in a electric field, the drift velocity is:

$$\begin{cases} v_x = -\dfrac{e\tau}{m}E_x - \omega_c \tau v_y \\ v_y = -\dfrac{e\tau}{m}E_y + \omega_c \tau v_x \\ v_z = -\dfrac{e\tau}{m}E_z \end{cases}$$

$\omega_c = \dfrac{eB}{m}$ —cyclotron frequency

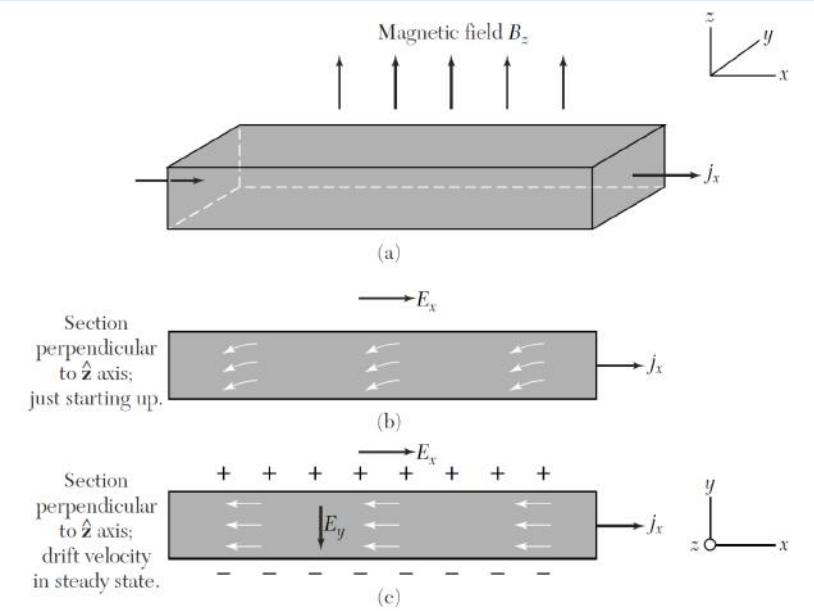
6.4 Hall Effect

$$\vec{J} = -ne\vec{v}$$

$$\sigma = ne^2\tau/m$$

$$\sigma_0 E_x = J_x + \omega_c \tau J_y$$

$$\sigma_0 E_y = -\omega_c \tau J_x + J_y$$



For a long rod, apply external elctric field E_x, there exists electric current J_x. Apply B field along z axis, the Lorentz force is along the negative y direction.

- ✓ **Electrons gather on the negative y side of the conductor, establishing a transverse field---Hall field E_y, which *cancels out* the Lorentz force, and the net current flows along x direction.**

6.4 Hall Effect

$J_y = 0$, **Hall electric field:**

$$E_y = -\frac{\omega_c \tau}{\sigma_0} J_x = -\frac{B}{ne} J_x$$

define $R_H = \dfrac{E_y}{J_x B}$ **--Hall coefficient**

$$\Rightarrow \quad R_H = -\frac{1}{ne}$$

Free electron approximation, Hall coefficient is negative.

Measuring Hall coefficient is one important tool for determining density of charge carriers.

6.4 Hall Effect

✓ **Quantum Hall effect**

For a two-dimensional electron system which can be produced in a MOSFET, in the presence of large magnetic field strength and low temperature, one can observe the quantum Hall effect, which is the ***quantization of the Hall voltage***.

✓ **Spin Hall effect**

It was predicted by M. I. Dyakonov and V. I. Perel in 1971 and observed experimentally more than 30 years later, both in semiconductors and in metals, at cryogenic as well as at room temperatures.

✓ **Quantum spin Hall effect**

✓ **Anomalous Hall effect**

6.5 Thermal Conductivity of Metals

thermal current : $J_Q = -\kappa \nabla T$

thermal conductivity : $\kappa = \kappa_e + \kappa_{Ph}$

insulator — phonon

metal — electron

physical picture:

Electrons have higher kinetic energy around the hot end, and diffuss to the cold end; Electrons have higher electric potential energy in the cold end, and diffuse to the hot end.

There exist *energy flow*, but *no* net electric current.

6.5 Thermal Conductivity of Metals

Semiclassical: electrons near the Fermi surface are responsible for the thermal conductivity.

free electron gas:

$$\kappa = \frac{1}{3} c_V v l \qquad \boxed{c_V = \frac{\pi^2}{2} n k_B \frac{T}{T_F}}$$

$$\Rightarrow \quad \kappa_e = \frac{1}{3}\left(\frac{\pi^2 n k_B T}{2 T_F}\right) v_F l_F$$

$$T_F = \frac{\varepsilon_F}{k_B} = \frac{1}{2}\frac{m v_F^2}{k_B} \qquad \frac{l_F}{v_F} = \tau_F$$

$$\Rightarrow \quad \boxed{\kappa_e = \frac{\pi^2 k_B^2 n \tau_F}{3m} T}$$

6.5 Thermal Conductivity of Metals

Wiedemann-Franz Law (1953) :

At not too low temperatures, the ratio of thermal conductivity to electric conductivity is proportional to temperature T, *i.e.*, $\frac{\kappa}{\sigma} \propto T$ **.**

electric conductivity : $\sigma = \frac{ne^2\tau_F}{m}$ $\qquad \kappa_e = \frac{\pi^2 k_B^2 n \tau_F}{3m} T$

$$\frac{\kappa_e}{T\sigma} = \frac{\pi^2}{3}\left(\frac{k_B}{e}\right)^2 \equiv L$$

***L*--Lorentz number**

$k_B = 1.38064852(79) \times 10-23$ J/K
$e = -1.602176565(35) \times 10-19$ C

$$L = 2.45 \times 10^{-8}\ \mathrm{W \cdot \Omega / K^2}$$

$$\left(\frac{J}{C}\right)^2 = V^2 = W \cdot \Omega$$

6.6 Failures of the free electron gas model

I. Drude used classical statistical mechanics, corrected by Sommerfeld by applying Fermi-Dirac statistics.

Successfully Accounts for a wide range of metallic properties.

II. Difficulties with the Free Electron Model

Inadequacies in the free electron model:

1. Static Thermodynamic Predictions: linear term prediction in the specific heat is very poor for transistion metals like iron.

2. Electron Transport Coefficients: Hall coefficient (-1/ne), magnetorisistence (independent of field strength), and others.

6.6 Failures of the free electron gas model

III. Review of Basic Assumptions

- Free Electron Approximation: ions play little role
- Independent Elec Approx.: elec-elec interactions are ignored
- Relaxation-Time Approx.: electrons have no "memory", collision happens on a average time of τ

Free Electron Approx. **is mainly responsible for the difficulties in Drude's and Sommerfeld's Metallic Theories:**

- ***Effects of ions*** **on the dynamics of electrons (between collisions)**
- **The role ions play during collision is left *unspecified***
- **Ions (independent dynamic entities) may also contribute significantly to physical properties (phonons in metals!)**